Only time will tell

Risk optimization from a dynamics perspective

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Name dropping:
Many thanks to Alex Adamou, Bill Klein, Reuben Hersh, Murray Gell-Mann, Ken Arrow.

My perspective

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- 17th century: mainstream economics went down a dead-end.

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- 19th – 21st centuries: relevant mathematics developed.

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- 19^{th} – 21^{st} centuries: relevant mathematics developed.

Program

Re-derive formal economics from modern starting point.

Thesis:

Problem with randomness, *i.e.* risk.

17$^{\text{th}}$-century key concept $\rightarrow$ **parallel worlds**.

21$^{\text{st}}$-century mathematics $\rightarrow$ **time**.

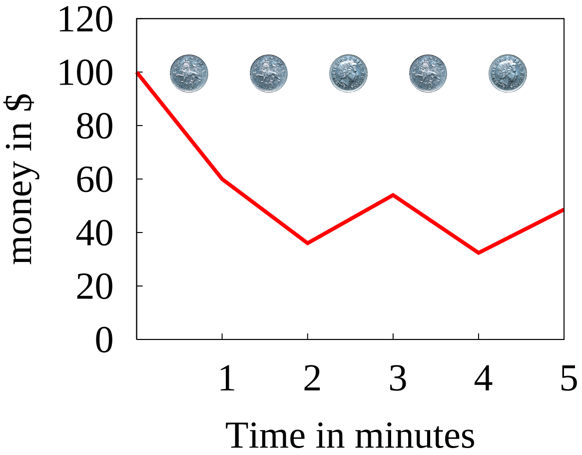
Innocuous game

Heads: win 50%.

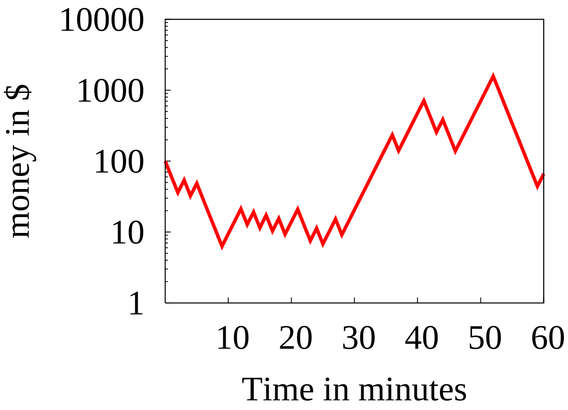
Tails: lose 40%.

Innocuous game

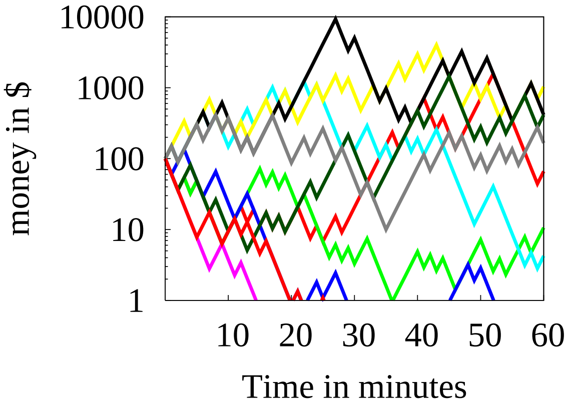
Toss coin once a minute



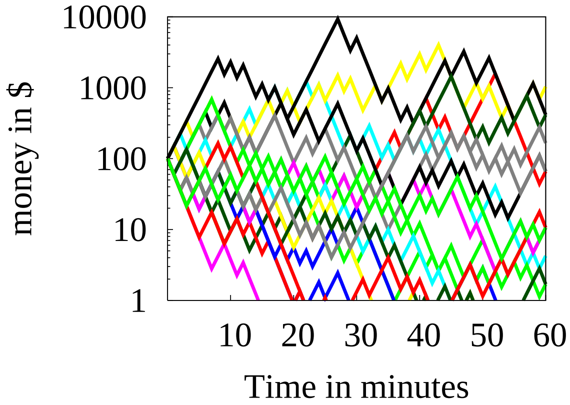
One sequence



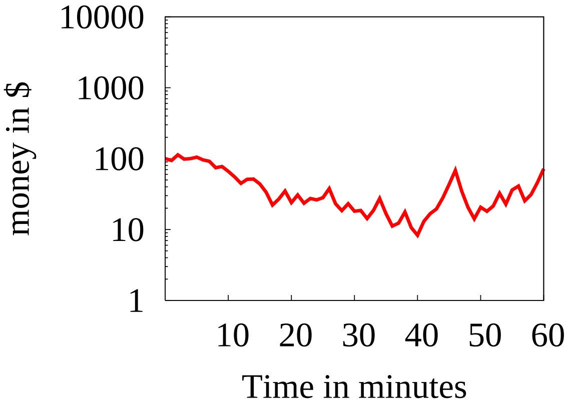
10 sequences



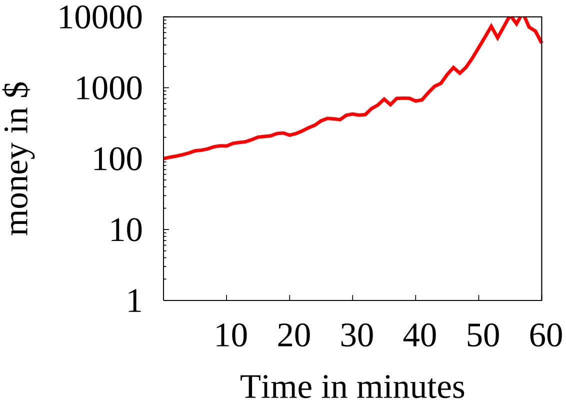
20 sequences



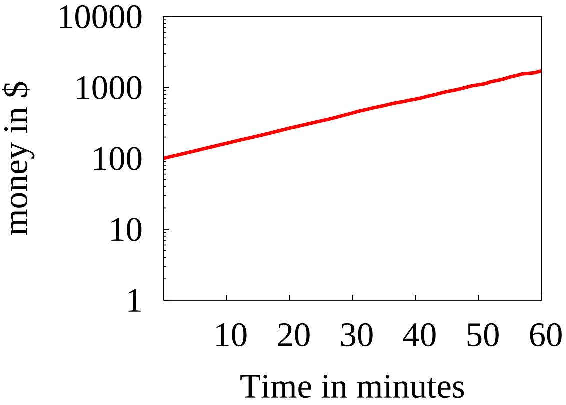
Average of 20 sequences



Average of 1000 sequences

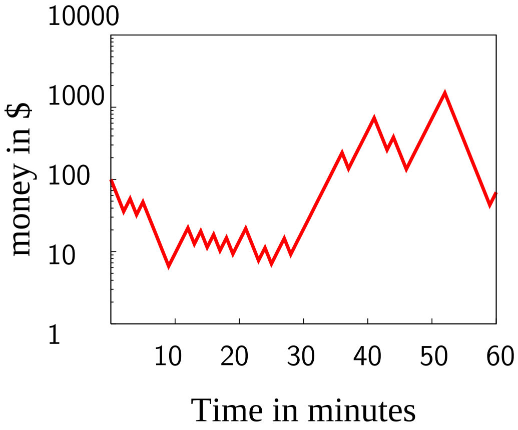


Average of 1,000,000 sequences

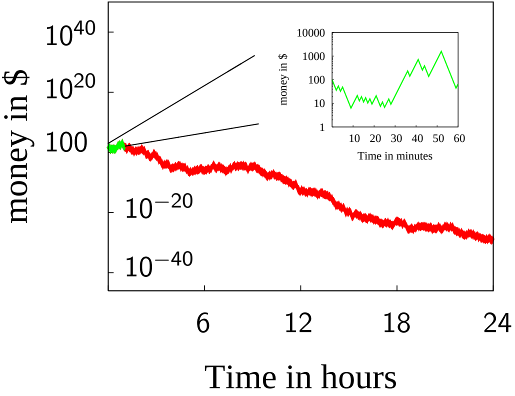


Good game?

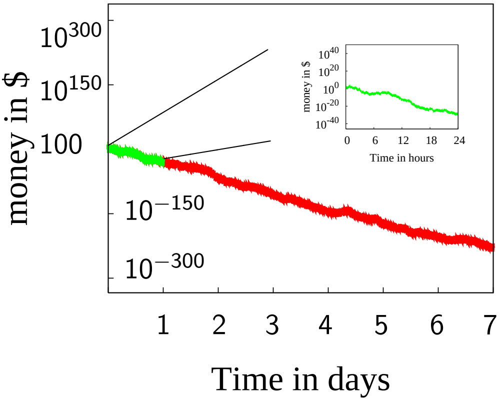
Play for one hour...



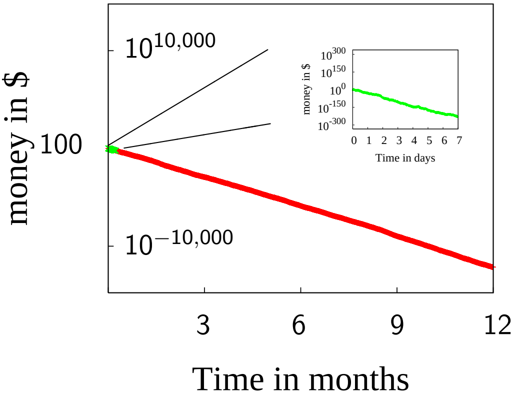
..continue one day (note scales)...



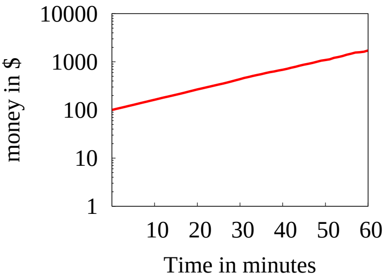
..continue one week (note scales)...



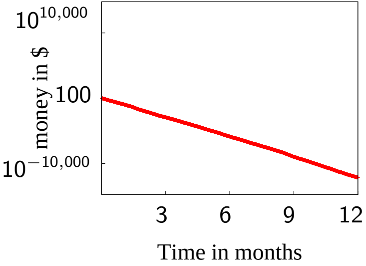
..continue one year (note scales)...



Ensemble perspective

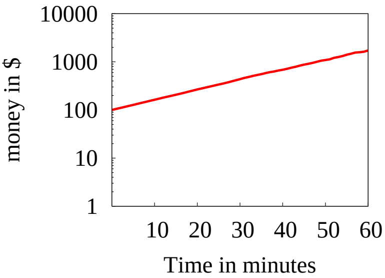


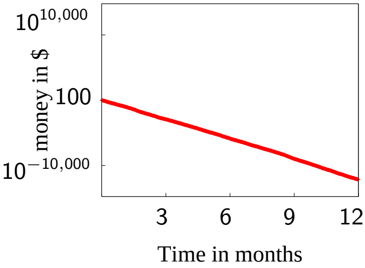
Time perspective



Non-ergodic

Ensemble perspective $\neq$ Time perspective





Non-ergodic

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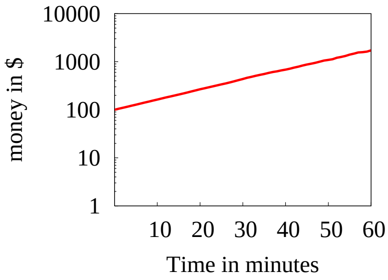
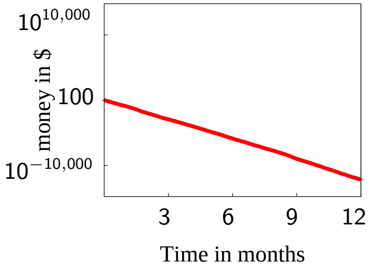


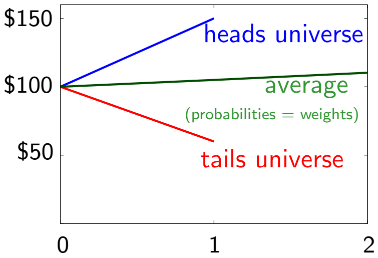


Non-commuting limits

$$\lim_{T\to\infty} \lim_{N\to\infty} g_{\text{est}} \neq \lim_{N\to\infty} \lim_{T\to\infty} g_{\text{est}}$$

No magic.

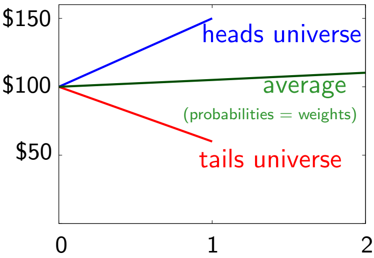
Ensemble perspective



Result: $110.25

No magic.

Ensemble perspective



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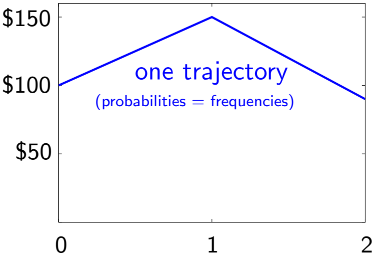
Time perspective



Result: $90

Message:

Expectation value meaningful only if

- observable is ergodic
- a physical ensemble exists

Otherwise meaningless.

Leverage optimization

Problem: find proportion of wealth to invest in some venture.

	Neoclassical economics	**Time perspective**
Model	Random variable Δx to represent changes in wealth.	Stochastic process $x(t)$ to represent wealth over time.
Technique	1656 –1738: compute expectation value $\langle \Delta x \rangle$. 1738 onwards: find utility function $u(x)$, optimize expectation value $\langle \Delta u(x) \rangle$.	Find ergodic observable $f(x)$. Optimize time-average performance by computing expectation value of ergodic observable.

Example:
geometric Brownian motion (GBM) with leverage l.

- Proportion l invested in GBM
- Proportion $1 - l$ invested in risk-free asset
- constant rebalancing, self-financed portfolio.

Wealth follows: $dx = x((\mu_r + l\mu_e)dt + l\sigma dW)$

$$\text{Solution: } x(t) = x_0 \exp\left(\left(\mu_r + l\mu_e - \frac{l^2\sigma^2}{2}\right)t + l\sigma W(t)\right)$$

→ Find optimal leverage using

a) Utility theory.

b) Time perspective.

a) Utility theory with power-law utility, $u(x) = x^\alpha$:

- Fix horizon Δt, consider random variable $x(\Delta t)$.
- Convert $x(\Delta t)$ to utility $u(x(\Delta t)) = x(\Delta t)^\alpha$,
  $u(x(\Delta t)) = x_0^\alpha \exp\left(\alpha\left(\mu_r + l\mu_e - \frac{l^2\sigma^2}{2}\right)\Delta t + \alpha l \sigma W(\Delta t)\right)$
- Find expectation value of $u(x(\Delta t))$,
  $\langle u(x(\Delta t))\rangle = x_0^\alpha \exp\left(\alpha \Delta t\left(\mu_r + l\mu_e - \frac{l^2\sigma^2}{2} + \frac{\alpha l^2\sigma^2}{2}\right)\right)$

  Implies expected change in utility $\langle \Delta u\rangle = \langle u(x(\Delta t))\rangle - u(x_0)$.
- Set derivative to zero,
  $\frac{d\langle \Delta u\rangle}{dl} = 0$
  $= x_0^\alpha \alpha \Delta t\left(\mu_e - l\sigma^2 + \alpha l \sigma^2\right)\exp\left(\alpha\Delta t\left(\mu_r + l\mu_e - \frac{l^2\sigma^2}{2} + \frac{\alpha l^2 \sigma^2}{2}\right)\right).$
- Solve for l

$$l_{\text{opt}}^{u} = \frac{\mu_e}{(1-\alpha)\sigma^2}.$$

b) Time perspective

- Given $x(t)$, find ergodic observable, *i.e.* $f(x)$ such that

$$\overbrace{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x(t))dt}^{\text{Time average}} = \overbrace{\frac{1}{N}\sum_i^N f(x_i(t)) = \langle f(x(t))\rangle}^{\text{Ensemble average}}.$$

Solution is a growth rate, determined by dynamics, $f(x) = \frac{1}{\Delta t}\log(x(t+\Delta t)/x(t))$.

- Find time average (or expectation value)

$\overline{f} = \mu_r + l\mu_e - \frac{l^2\sigma^2}{2}$.

- Set derivative to zero, solve for l

$$\underline{l^t_{\text{opt}} = \frac{\mu_e}{\sigma^2}.}$$

Comments:

- $l^u_{\text{opt}} = \frac{\mu_e}{(1-\alpha)\sigma^2}$ depends on utility function (α).

  $l^t_{\text{opt}} = \frac{\mu_e}{\sigma^2}$ set by dynamics.

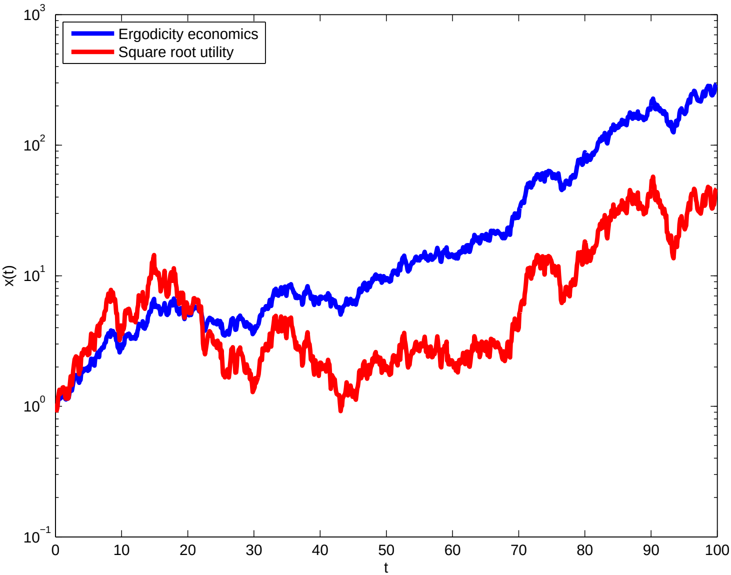
- Utility: 18$^{\text{th}}$ century (long before ergodicity debate).

- Modern interpretation
  Utility theory aims for ergodic observable, *e.g.* for GBM, rate of change in log utility.

- Different questions answered
  Utility theory:
  l^u_{opt} corresponds to greatest expected happiness.

  Time perspective:
  l^t_{opt} implies greatest growth rate.

GBM parameters $\mu_r = 5.2\%$ p.a., $\mu_e = 2.4\%$ p.a., $\sigma = 15.9\%$ p.$\sqrt{a.}$



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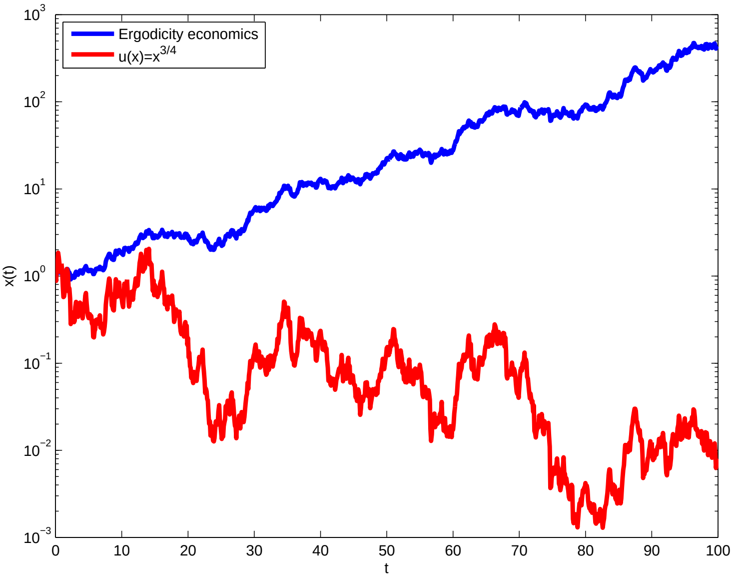

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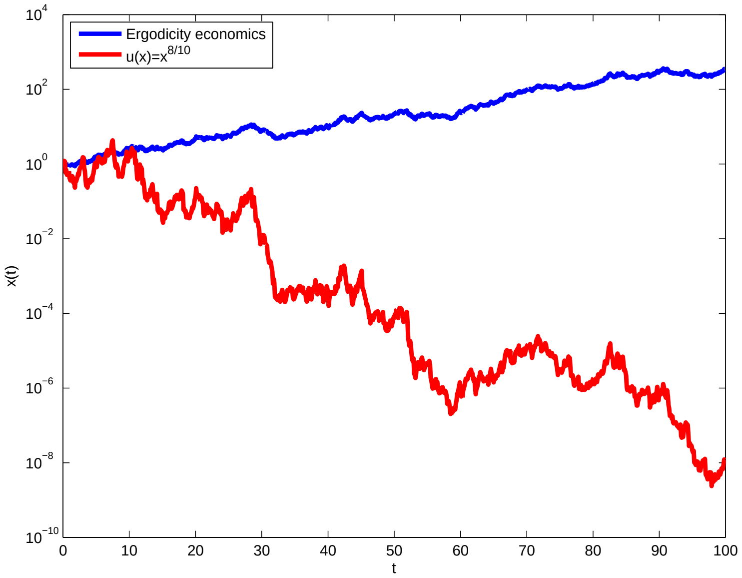
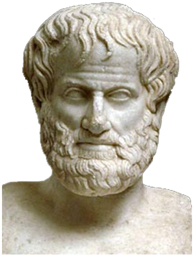


Cruel world doesn't care about my risk preferences.

How deep the rabbit hole goes

384 – 322 BC Aristotle's cosmology.



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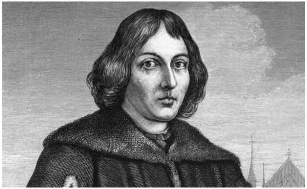
310 – 230 BC	Aristarchus model: heliocentric – dismissed.



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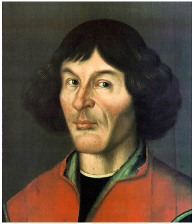

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Mid-17$^{\text{th}}$ century: Crisis! All or nothing time-bound?

1654 Probability theory

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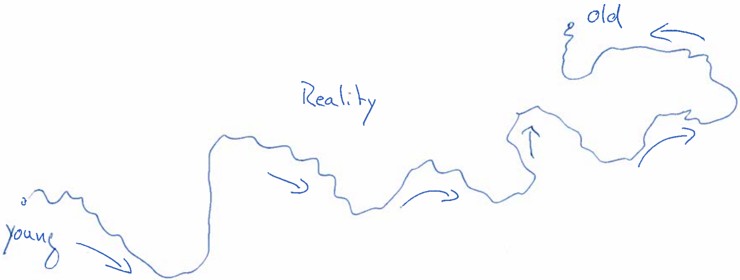

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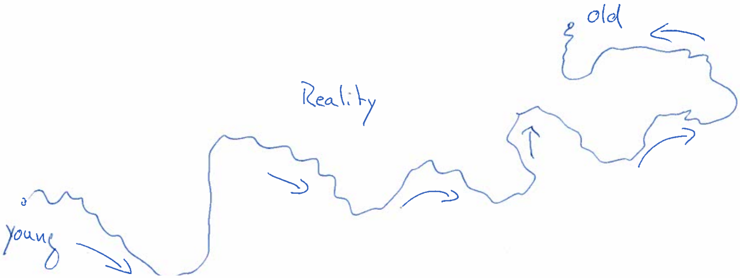

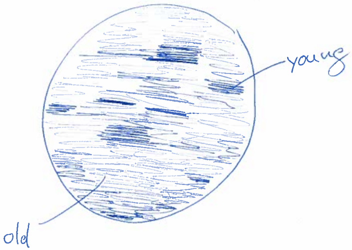

Evolution of structure

Two entities follow GBM.

$dx_1 = x_1(\mu dt + \sigma dW_1)$ and $dx_2 = x_2(\mu dt + \sigma dW_2)$

If entities pool resources, they will follow

$$dx_{12} = x_{12}\left[\mu dt + \sigma\left(\frac{1}{2}dW_1 + \frac{1}{2}dW_2\right)\right]$$

Cooperation conundrum

Lucky partner gives, unlucky partner receives.

Expectation value grows at μ, irrespective of cooperation.

Lucky partner: why cooperate?

But: cooperation exists (multicellularity, firms, states etc).

$\rightarrow$ **Expectation value model fails.**

Usual story: very complicated.

Our story: non-ergodic system $\rightarrow$ compute expectation value *of ergodic observable* under specified dynamics (time-average growth rate).

1. No cooperation: $\frac{d\langle \ln(x_1)\rangle}{dt} = \frac{d\langle \ln(x_2)\rangle}{dt} = \mu - \sigma^2/2$
2. Cooperation: $\frac{d\langle \ln(x_{12})\rangle}{dt} = \mu - \sigma^2/4$

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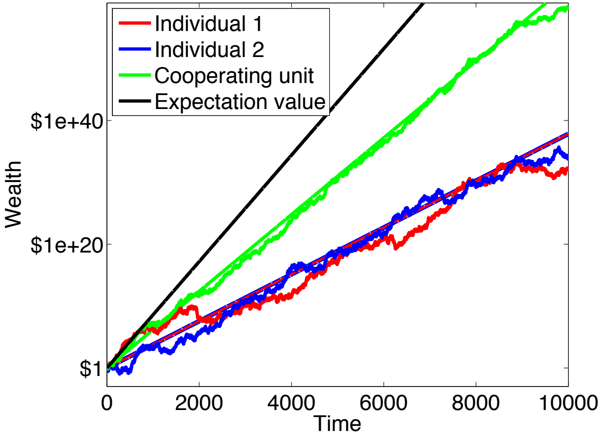
2. Cooperation: $\frac{d\langle \ln(x_{12})\rangle}{dt} = \mu - \sigma^2/4$

   Cooperators do better *over time* (though not in expectation).

von Neumann (1944):
"We need inventions on the scale of a new calculus to make progress on dynamics."

Correct, and we have that since 1944 (Itô).



- Good risk management = faster growth.
- Evolutionary advantage of structure (multicellularity, tribes, firms, nations) over no structure.

Problems we can address (solve)

- optimize leverage (for any dynamic)
- map utility theory $\rightarrow$ dynamics
- 300-year old St. Petersburg paradox
- dynamics of wealth distribution
- better economic measures than GDP
- price insurance contracts (derivatives)
- explain emergence of structure
- ...

Conclusion

Correcting one deep conceptual flaw in economics enables powerful quantitative theory.

Thank you.